Thermodynamics of Strings and Hagedorn Temperature

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This article deals with the thermodynamics of the strings and Hagedorn temperature. I will calculate the partition function of the strings and Hagedorn temperature and use the idea of statistical mechanics to derive entropy of black holes.

1. INTRODUCTION:

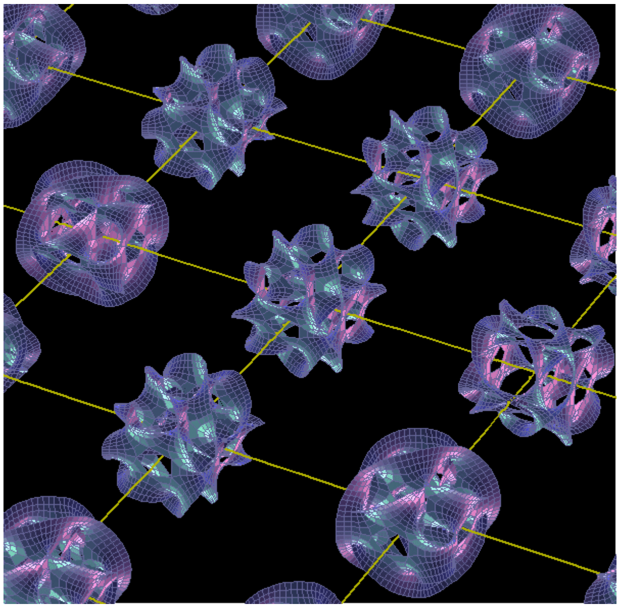

Figure 1: String theory

String theory is one of the most exciting theories of research now-a-days. This ambitious and speculative theory offers the potential of unifying gravity and all the other forces of nature and all forms of matter into one unified conceptual structure. The thermodynamics of strings is governed largely by the exponential growth of the number of quantum states as a function of its energy. To find partition function of string we can approach the problem as a collection of harmonic oscillators. We will see how a relativistic string at high energies appears to approach a constant temperature known as the Hagedorn temperature. This is the temperature above which partition sum diverges in a system with exponential growth in the density of the states. And in the end we will use our results from string theory and ideas of thermodynamics to derive entropy of black holes, another very interesting topic.

2. STEPS TO BE FOLLOWED:

To analyze the strings thermodynamics and find Hegedorn temperature, I would follow following steps:

- Calculate number of partitions of large integers analyzing a quantum non-relativistic string or “quantum violin string” at high temperature.
- Calculate entropy-energy relation for an idealized quantum relativistic string.
- Finding Hagedorn temperature.
- Assemble results to compute the partition function of the relativistic string.
- Analyze black hole thermodynamics.

3. QUANTUM VIOLIN STRING AND NUMBER OF PARTITIONS:

Consider a quantum violin string which is defined as quantum mechanical non-relativistic string with fixed end-points. Such type of string has an infinite set of vibrating frequencies, all multiples of some basic frequency ω_o. Let’s idealize this quantum string as a collection of simple harmonic oscillators with frequencies ω_o, $2\omega_o$, $3\omega_o$ and so on. We know that Hamiltonian of a simple harmonic oscillator is given by:

$$\hat{\mathrm{H}} = \hbar\omega\left(a^{\dagger}a + \frac{1}{2}\right) \qquad (1)$$

Where $a^{\dagger}$ is creation operator and a is annihilation operator. If we discard zero-point energy each simple harmonic oscillator has its own creation and annihilation operator and hamiltonian which can be written as follows:

$$\hat{\mathrm{H}}_{\omega_o} = \hbar\omega_o a_1^{\dagger}a_1 \qquad (2)$$

$$\hat{\mathrm{H}}_{2\omega_o} = 2\hbar\omega_o a_2^{\dagger}a_2 \qquad (3)$$

$$\hat{\mathrm{H}}_{3\omega_o} = 3\hbar\omega_o a_3^{\dagger}a_3 \qquad (4)$$

$$\dots$$

Where the conventional commutation relation is satisfied,

$$\left[a_m, a_n^{\dagger}\right] = \delta_{mn} \qquad (5)$$

Since, our quantum string is union of all these oscillators, Hamiltonian $\widehat{H}$ will be:

$$\hat{\mathrm{H}} = \sum_{l=1}^{\infty} H_{l\omega_o} \qquad (6)$$

$$\hat{\mathrm{H}} = \hbar\omega_o \sum_{l=1}^{\infty} l a_l^{\dagger}a_l \qquad (7)$$

Where we can call $\sum_{l=1}^{\infty} l a_l^{\dagger} a_l$= N as number operator. It's an observable that counts the number of particles. So,

$$\hat{\mathrm{H}} = \hbar\omega_o \hat{N} \tag{8}$$

Now, the vacuum state or state of lowest possible energy of the string is |Ω> such that a_l|Ω> = 0 for all *l*

A quantum state |Ω> of this string is obtained when creation operators act on the vacuum.

$$|\Psi> = (a_1^{\dagger})^{n_1} (a_2^{\dagger})^{n_2} \ldots (a_l^{\dagger})^{n_l} \ldots |\Omega> \tag{9}$$

So, the occupation number state is defined by is given by set $\{n_1, n_2, \ldots\}$

The number operator has an eigenvalue N which is:

$$N = n_1 + 2n_2 + 3n_3 + \cdots \tag{10}$$

$$N = \sum_{l=1}^{\infty} l n_l \tag{11}$$

Then the energy is given by:

$$E = \hbar\omega_o N \tag{12}$$

Until now it was pretty straight forward. Now we will deal with the real question that is for a fixed positive integer N, how many states are there with $\hat{N}$ eigenvalue equal to N. Let us denote this number with p(N) and call it *number of partitions of N.* A partition of *N* is a set of positive integers that add up to *N.*

Instead of finding p(N) which would be difficult, we would find $\ln p(N)$ accurately for large *N.*

We know that entropy S of a system is defined in terms of the number of states $\Omega(E)$ as:

$$S(E) = k ln\, \Omega(E) \tag{13}$$

For a fixed *E*,

$$N = \frac{E}{\hbar\omega_o} \tag{14}$$

And $\Omega(E)$ is simply $p(N)$. So, entropy becomes:

$$S(E) = k ln\, p\left(\frac{E}{\hbar\omega_o}\right) \tag{15}$$

So, now our motive is to find $S(E)$. For that we will calculate partition function of quantum violin string.

4. PARTITION FUNCTION, FREE ENERGY AND ENTROPY:

We know partition function is given by:

$$Z = \sum_{\alpha} exp\left(-\frac{E_\alpha}{kT}\right) \qquad (16)$$

Putting the value of Energy:

$$Z = \sum_{n_1, n_2, n_3, \ldots} \exp\left[-\frac{\hbar\omega_o}{kT}(n_1 + 2n_2 + 3n_3 + \cdots)\right] \qquad (17)$$

Now, we see that the set of all states is labeled by set of occupation numbers. So, to sum over all states, we sum over all occupation numbers ranging from 0 to infinity and since exponentials of a sum can be written as product of exponentials, Z becomes,

$$Z = \prod_{l=1}^{\infty} \sum_{n_l=0}^{\infty} \exp\left(-\frac{\hbar\omega_o l n_l}{kT}\right) = \prod_{l=1}^{\infty}\left[1 - \exp\left(-\frac{\hbar\omega_o l}{kT}\right)\right]^{-1} \qquad (18)$$

Where we used geometric series to get the result.

Next step is to find free energy F which can be found by the formula:

$$F = -kT \ln Z = -kT \sum_{l=1}^{\infty} \ln\left[1 - \exp\left(-\frac{\hbar\omega_o l}{kT}\right)\right] \qquad (19)$$

Here we are make some assumptions. We can only change this summation to integral if we assume Temperature is high enough such that

$$\frac{\hbar\omega_o}{kT} < 1$$

Then, each term in the summation differs very little than the previous one and we get the integral.

$$F \approx -kT \int_{1}^{\infty} \ln\left[1 - exp\left(-\frac{\hbar\omega_o l}{kT}\right)\right] \qquad (20)$$

Using the expansion:

$$\ln(1-y) = -\left(y + \frac{1}{2}y^2 + \frac{1}{3}y^3 + \cdots\right) \tag{21}$$

And

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \zeta(2) = \frac{\pi^2}{6} \tag{22}$$

And then we finally obtain high temperature approximation for free energy as

$$F \approx -\frac{(kT)^2}{\hbar\omega_o}\left(\frac{\pi^2}{6}\right) = -\frac{\pi^2}{6\hbar\omega_o\beta^2} \tag{23}$$

And for this string, the free energy has no volume dependence.

As a last step, we will calculate entropy as a function of temperature which comes to be:

$$S = -\frac{\partial F}{\partial T} = \frac{\pi^2 k^2 T}{3\hbar\omega_o} \tag{24}$$

And computing energy since we are interested in finding entropy as a function of energy,

$$E = -\frac{\partial lnZ}{\partial \beta} = \frac{\pi^2 k^2 T^2}{6\hbar\omega_o} \tag{25}$$

So, we finally read off

$$p(N) = 2\pi\sqrt{\frac{N}{6}} \tag{26}$$

Which is our first goal.

5. GENERALIZING OUR RESULT FOR THE VIBRATING STRING:

If the string is vibrating in b transverse directions, then for each frequency $l\omega_o$, we have b harmonic oscillators that represent the possible polarizations of the motion. Now, in order to sum over all possible states in the new partition function Z_b, we must sum over all possible values of the occupation number n_k^q , where k= 1,2,…,∞ and q=1,2,…,b

This gives us, $Z_b = (Z)^b$

Using earlier result for $S(E)$, from equation (13) we have,

$$S_b = 2\pi k\sqrt{\frac{Nb}{6}} \qquad (27)$$

And in this case,

$$p_b(N) = 2\pi\sqrt{\frac{Nb}{6}} \qquad (28)$$

Obviously, if b=1 the expression reduces to previous expression (26).

6. HAGEDORN TEMPERATURE:

Now we are considering open strings that carry no spatial momentum. For example let's say open string end-points end on a D0-brane. When momentum is zero, the string has energy levels that are simply given by the rest masses of its quantum states.

As we know in a general case total energy E is,

$$M^2 + p^2 = E^2 \qquad (29)$$

When p=o, M=E

And number operator $N^{\perp}$ is given by:

$$N^{\perp} = \sum_{n=1}^{\infty} n a_n^{I\,\dagger} n a_n^I \qquad (30)$$

So, mass squared of a given state can be expressed in terms of the number operator

$$M^2 = \frac{1}{\acute{\alpha}}(-1 + N^{\perp}) \approx \frac{N^{\perp}}{\acute{\alpha}} \qquad (31)$$

In the approximation that $N^{\perp}$ is very large.

So, relation of energy to number operator is simple,

$$\sqrt{N^{\perp}} = \sqrt{\acute{\alpha}}E \qquad (32)$$

In the microcanonical ensemble, the number of states Ω(E) equals $p_I(N^{\perp})$ because we have I transverse light-cone directions. So, from equation (27)

$$S(E) = k\ 2\pi \sqrt{\frac{N^{\perp} I}{6}} \tag{33}$$

And in terms of energy from (32):

$$S(E) = k\ 2\pi \sqrt{\frac{\acute{\alpha}}{6} E} \tag{34}$$

When energy is large, entropy is also large. An Entropy proportional to the energy is unusual because it leads to a constant temperature. This temperature is called Hagedorn temperature. We know that:

$$\frac{\partial S}{\partial E} = \frac{1}{T} \tag{35}$$

So, from equation (34), we get

$$\frac{1}{T} = k\ 2\pi \sqrt{\frac{\acute{\alpha}}{6}} \tag{36}$$

This temperature is called Hagedorn temperature. Rearranging the expression for T_H

We get,

$$T_H = \frac{1}{k\ 2\pi} \sqrt{\frac{6}{\acute{\alpha}}} \tag{37}$$

In the high energy approximation, we can arbitrarily increase the energy of the strings and their temperature will remain fixed at Hagedorn Temperature.

7. SINGLE STRING PARTITION FUNCTION:

We know that a particle confined in a box of volume V has its energy and momentum dependence as:

$$E(p) = \sqrt{p^2 + m^2} \tag{38}$$

And the partition function is given by:

$$Z(m^2) = \sum_p \exp\left(-\beta E(p)\right) \tag{39}$$

This is a relativistic particle of mass m that lives in D-dimensional space time or $d = D - 1$

Quantized values of momenta depend on the dimensions of the box. That is why partition function has volume dependence. We will sum over all the momentas and approximate it by an integral, then we get

$$Z(m^2) = V \int \frac{d^d p}{(2\pi\hbar)^d} \exp\left(-\beta\sqrt{p^2 + m^2}\right) \tag{40}$$

This is the integral representation of the partition function for a relativistic point particle of rest mass m. To further simplify the integral we take into account the fact that at temperatures well below Hagedorn temperature, thermal energy of all strings is less than their rest mass energy. This transforms the integral to simple Gaussian which is evaluated to get following result.

$$Z(m^2) = \ V \exp(-\beta m) \left(\frac{m}{2\pi\beta}\right)^{\frac{d}{2}} \tag{41}$$

This is the partition function of a relativistic particle in the low temperature limit.

Now to find the partition function of a string we will consider a string placed in a box of volume V. Total quantum states available to it can be obtained by writing a set of basis states as in equation,

$$|\lambda, p> = \prod_{n=1}^{\infty} \prod_{I} (a_n^{I\dagger})^{\lambda_{n,I}} |p^+, p_T> \tag{42}$$

Where $p_T$is the transverse momentum, components of which are eigenvalues of p^Ioperator.

And

$$M^2(\{\lambda_{n,I}\}) = \frac{1}{\alpha}(N^{\perp} - 1) \tag{43}$$

Each string state is represented with a set of occupation number and spatial momentum. So, energy of each state can be written as

$$E(\{\lambda_{n,I}\},p) = \sqrt{M^2(\{\lambda_{n,I}\})+p^2} \tag{44}$$

And for finding the partition function, we have to sum over all states,

$$Z_{str} = \sum_{\alpha} \exp\left(-\beta E_{\alpha}\right) = \sum_{\lambda_{n,I}} \sum_{\beta} \exp\left[-\beta\sqrt{M^2(\{\lambda_{n,I}\})+p^2}\right] = \sum_{\lambda_{n,I}} Z(M^2(\{\lambda_{n,I}\})) \tag{45}$$

Since, M^2 only depends on $N^{\perp}$, which is equal to N, then we can write as:

$$Z_{str} = \sum_{N=0}^{\infty} p_I(N) Z(M^2(N)) \qquad (46)$$

Let us re-write Z_{str} as a sum where N_0 denotes the integer for which $p_I(N)$ is known by equation (28).

$$Z_{str} = Z_0 + \sum_{N=N_0}^{\infty} p_I(N) Z(M^2(N)) \qquad (47)$$

Where summation of Z_0 goes from 0 to $(N_0 - 1)$.

When temperature approaches Hagedorn temperature, Z_0 becomes negligible and second term on the right hand side becomes very large.

Changing summation to integral,

$$Z_{str} \cong Z_0 + \int_{N_0}^{\infty} p_I(N) Z(M^2(N)) \qquad (48)$$

Now, if we define density of states (M) as a function of mass M, use the mass as a variable of integration and put value of Z, the equation becomes:

$$Z_{str} \cong Z_0 + \sqrt{2} \int_{M_0}^{\infty} \left(\sqrt{\acute{\alpha}} M\right)^{-\frac{I}{2}} \exp(\beta M)\, Z(M^2)\, d\left(\sqrt{\acute{\alpha}} M\right) \qquad (49)$$

We just need to put the value of $Z(M^2)$ from equation (39). It gives Z in terms of M and kT_H and simplify it further to get the partition function of string:

$$Z_{str} \cong Z_0 + \frac{2^{11}}{\pi} V\, (kT\, kT_H)^{\frac{I}{2}} \left(\frac{T}{T_H - T}\right) \exp\left(-4\pi\sqrt{N_0}\, \left[\frac{T_H}{T} - 1\right]\right) \qquad (50)$$

When T goes to T_H, argument of exponent goes to 0 and as a result exponential goes to one and the factor that multiplies becomes much larger than Z_0. For T sufficiently close to 0, partition function becomes:

$$Z_{str} \cong \frac{2^{11}}{\pi} V\, (kT_H)^{\frac{I}{2}} \left(\frac{T}{T_H - T}\right), \quad T \to T_H \qquad (51)$$

This is our required expression for the approximate partition function of a single open string that is enclosed in volume V and in thermal contact with a temperature close to T_H.

8. BLACK HOLES AND THEIR ENTROPY:

A black hole is gravitationally collapsed object that is formed when mass of an object is increased while its size remains fixed or when size of an object is reduced while the mass is kept constant. The simplest black hole 'Schwartschild black hole' has its two main components. Singularity and an event horizon, Singularity is what is left of the collapsed star and its theoretically a point of zero dimension, infinite density and a finite mass and event horizon is region of space that is boundary of black hole. Past this point, nothing can escape. A significant success of string theory is the statistical mechanical derivation of the entropy of black holes.

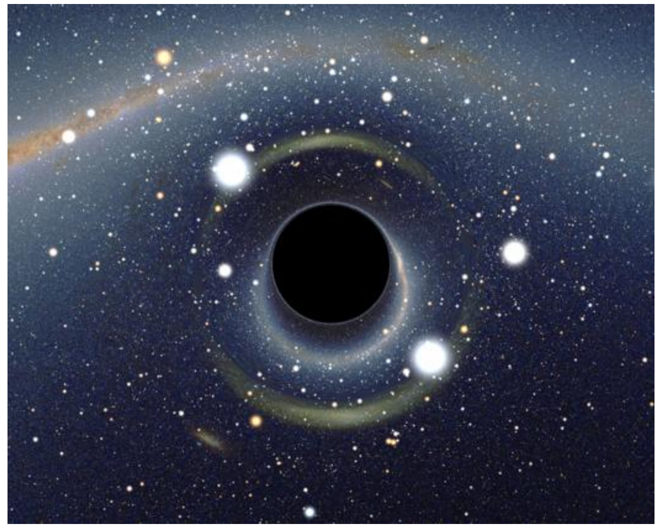

Figure 2: A Black Hole

Assume R to be radius of a Schwartschild black hole. Its value can be estimated by assuming that the total energy of any particle at the horizon is equal to zero. If particle has a mass *m* this energy includes rest mass energy and gravitational potential energy and we will set the sum to be zero.

$$mc^2 - \frac{GMm}{R} = 0 \qquad (52)$$

Which means that,

$$R = \frac{GM}{c^2} \qquad (53)$$

The interesting fact is that Schwartschild radius of sun is about 3 kilometers and that of earth is about one centimeter.

We can also estimate gravitational field at the horizon using Newtonians law of gravitation,

$$g = \frac{GM}{R^2} = \frac{c^4}{4GM} \qquad (54)$$

Now assume that a certain amount of hot gas falls into a black hole such that now its mass is slightly increased. By second law of thermodynamics, total entropy of a system which is now consisted of gas and black hole should not decrease. The more general statement would be when common entropy goes down a black hole, the common entropy of the black hole exterior plus the black hole entropy never decreases. So, black hole should acquire at least as much entropy as much carried by gas. Hence, we conclude that black holes have entropy.

Black holes emit radiations with a well-defined temperature, known as Hawking temperature $T_{H'}$, which is given by:

$$T_{H'} = \frac{\hbar c^3}{8\pi GMk} \quad (55)$$

Using $E = mc^2$ for the energy of black hole and first law of thermodynamics, we can calculate entropy using equation.

$$dE = T_{H'} dS \quad (56)$$

And

$$dE = c^2 dM \quad (57)$$

Comparing these two equations putting value of $T_{H'}$ and solving for dS, we get

$$dS = \frac{4\pi G}{\hbar c} k \, dM^2 \quad (58)$$

Integrating it, assuming that entropy of a zero mass black hole is zero, we have

$$S = \frac{4\pi G}{\hbar c} k \, M^2 \quad (59)$$

In string theory, we attempt to relate a stationary black hole to a string with high degree of excitation but zero momentum. Entropy of the string by equation putting E=M,

$$S_{str} = 4\pi\sqrt{\alpha'} \, k \, M \quad (60)$$

Comparing equations and we see that entropy of a black hole varies like mass-squared and that of string goes like the mass.

These equations disagree because black hole entropy was calculated in a regime where interactions are necessary while the string entropy was calculated for free strings. If we do calculations using string interactions and string coupling constant, then these two equations show complete agreement.

Let us see how string theory estimates the entropy of a black hole. When strings fall onto a horizon an external observer sees them spread out and fill the stretch horizon. We can compare this phenomenon as melting of strings when they encounter Hagedorn temperature at some distance from the event horizon. The entropy of single string states is so large that strings on the horizon will tend to form a single string when the Hagedorn temperature is approached. We imply that all black hole states are in one to one correspondence with single string states.

Now, it has been observed in the past that the high mass-low angular momentum states of string theory must be black holes since they lie within their Schwarzshild radii. Let’s assume string coupling exists and is significant and consider radius of an average excited string ignoring all higher order effects including the long range gravitational field. Several things happen at this point. The first is that the area of the black hole horizon has become equal to length of the string. Second, the Hawking temperature has reached the Hagedorn temperature. Third, the conventional mass M and the stretched horizon energy become equal. Finally at this point the Black hole entropy becomes of order the string entropy.

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